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CREATIVE DESTRUCTION AND BUSINESS CYCLES

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Abstract

In a dynamic general equilibrium setup, this paper aims at providing a general framework for the analysis of the role of vintages and creative destruction on business fluctuations. By stressing the forward-looking behavior of the optimal scrapping rule, we use a standard rational expectations argument to show, in the linear utility case, the time independence of the scrapping function. Secondly, we prove that equilibrium output shows a purely periodic behavior around an exponential growth trend, the pattern of the cycle being determined by the pattern of initial conditions. The vintage capital model presented in this paper provides a new view on business fluctuations: historical conditions are at the basis of business fluctuations, in the sense that historically volatile or stable economies will reproduce their own historical pattern in the future.

Key Words

Business Cycles, Creative Destruction, Periodic Equilibria, Vintage Capital

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1 Introduction

In the sixties, there was an important development of the so-called "vintage capital" model. The boom of this literature was mainly due to some interesting differences with respect to the standard neoclassical growth model: technological progress is embodied in new equipment and obsolescence, the replacement of old machines, is an economic decision. This property of the vintage model allows it to formalize the Shumpeterean idea of *creative destruction*: more productive machines replace old ones. Technical problems related to the solution of these dynamic systems were at the basis of the decline of this approach. However, in the 90's some new researches have started. First, by analyzing the stationary state of general equilibrium economies with vintage capital (see for example Chari and Hopenhayn (1991) and Cooley, Greenwood and Yorukoglu (1994)). Secondly, Caballero and Hammour (1994), in a partial equilibrium economy, analyze the dynamic of scrapping and replacement under an exogenous demand process. Finally, Benhabib and Rustichini (1991) study the role of different *exogenous* depreciation schedules on growth.

This paper intends to provide a general framework for the analysis of the role of vintages on business cycles, by stressing the forward-looking behavior of the optimal scrapping rule. An interesting feature of the original "vintage model" is that the optimal scrapping policy depends on the expected scrapping of contemporaneous investment. The firm is choosing simultaneously to scrap old machines and to replace them by new ones, with expected benefits depending on their future scrapping. This problem takes the form of a rational expectations (perfect-foresight) problem. However, it is not standard since the resulting temporal leads are endogenous.

On the line of the production problem proposed by Malcomson (1975), we provide the solution for a dynamic general equilibrium economy with vintage capital. We limit ourselves to the linear utility case, which allows us to take advantage of the van Hilten (1991) result, to show that the scrapping rule is time independent. At equilibrium, as in the standard neoclassical growth model, the growth trend follows an exogenously given technical progress (population growth is supposed to be zero), and, depending on initial conditions given on a certain interval, there is a purely periodic behavior around this trend. The length of this interval is endogenously determined by expectations, through the forward-looking component of the scrapping rule. This economy is a good example of economies in which expectations and history are both important: depending on expectations firms choose the optimal scrapping rule, and business fluctuations are endogenously determined by initial conditions over an interval which length is given by the optimal scrapping rule.

Two main schools in modern macrotheory debate about the causes of business cycles. Real business cycle theorists, among others, stress the hypothesis that a random environment is at the basis of real world fluctuations. On the other extreme, non-linear dynamic theorists think that cycles are mainly due to non-linearities. The vintage capital model presented in this paper provides a different view on fluctuations: the time profile of initial conditions, over an interval of endogenous length, is at the basis of economic fluctuations. We argue that historical conditions (i.e. initial conditions on an interval of time) are at the basis of observed business cycles, in the sense that historically volatile or stable economies will reproduce their own historical pattern in the future.

This paper is organized as follows. Section 2 describes the economy. In Section 3 the equilibrium is analyzed and the main propositions are stated. Section 4 concludes.

2 The Economy

Population is constant. There is only one final good, which can be assigned to consumption or investment and plays the role of numeraire (the aggregate price index is unity for all period t). The final good is produced in a competitive market by mean of a constant return to scale technology, which is defined over a continuum of inputs in the interval $[0, 1]$. Inputs are produced by mean of a linear technology defined over vintage capital and an homogeneous labor. The inputs market is supposed monopolistically competitive to allow for a concave profit function in the inputs sector. Finally, the labor market is competitive.

Individual's Behavior

Let us assume that the representative household solves a standard intertemporal maximization problem with linear instantaneous utility function:

$$\max_{\{c(t),a(t)\}} \int_0^\infty c(t) \exp\{-\rho t\} \, \mathrm{d}t \tag{1}$$

subject to the budget constraint

$$\dot{a}(t) = w(t) + r(t)\, a(t) - c(t)$$

with initial wealth a_0. $c(t)$ and $a(t)$ represent per-capita consumption and wealth respectively. The interest rate $r(t)$ and the wage rate $w(t)$ are taken as given by the representative household. It is supposed that the time preference parameter ρ is positive. We normalize to unity the labor endowment of individuals, so that the per-capita labor supply is $l(t) = 1$, $\forall t \geq 0$.

Since the instantaneous utility function is linear, the interest rate must be constant at equilibrium: $r(t) = \rho$, $\forall t \geq 0$. Consumption is undetermined in the consumer problem, but it is determined at equilibrium as explained later.

Final Good

The final good is produced competitively and the representative final firm solves the following problem

$$\max_{\{y_i(t)\}} \; y(t) - \int_0^1 p_i(t) y_i(t)\, \mathrm{d}i, \tag{2}$$

where the per-capita production $y(t)$ is given by a CES production technology

$$y(t) = \left(\int_0^1 y_i(t)^{\frac{\epsilon-1}{\epsilon}}\, \mathrm{d}i \right)^{\frac{\epsilon}{\epsilon-1}}$$

defined over a continuoum of inputs $y_i(t)$ with $i \in [0,1]$. Prices $p_i(t)$ are taken as given by the representative final firm. It is supposed that the elasticity of substitution $\epsilon > 1$.

As in the standard monopolistic competition economy,[1] the corresponding inverse demand function takes the following functional form:

$$p_i(t) = \left(\frac{y_i(t)}{y(t)} \right)^{-\frac{1}{\epsilon}}.$$

Input Firm

The representative input firm, producing in a monopolistically competitive market, solves the following problem:

$$\max_{\{p_i(t), y_i(t), l_i(t), \imath_i(t), T_i(t)\}} \int_0^\infty \left[p_i(t) y_i(t) - w(t) l_i(t) - \imath_i(t) \right] R(t)\; \mathrm{d}t \tag{3}$$

subject to

$$y_i(t) = \int_{t-T_i(t)}^{t} \imath_i(\tau)\; \mathrm{d}\tau \tag{4}$$

$$l_i(t) = \int_{t-T_i(t)}^{t} q(\tau)\, \imath_i(\tau)\; \mathrm{d}\tau \tag{5}$$

$$p_i(t) = \left(\frac{y_i(t)}{y(t)} \right)^{-\frac{1}{\epsilon}} \tag{6}$$

$$q(t) = q_0 \exp\{-\gamma t\}$$

with initial conditions

$$\imath_i(t) \;\text{ given } \quad \forall\; t < 0.$$

Variables $y(t)$, and $w(t)$ are, $\forall\, t \geq 0$, taken as given by the monopoly. Parameter γ and q_0 are both strictly positive. The discount factor takes the following form:

$$R(t) = \exp\{-\rho t\}.$$

[1] Dixit and Stiglitz (1977)

Following Malcomson (1975), the problem can be rewritten as

$$\max_{\{y_i(t),\imath_i(t),J_i(t)\}} \quad \int_0^\infty (y(t)^{\frac{1}{\epsilon}} y_i(t)^{1-\frac{1}{\epsilon}} - \phi_i(t)\, y_i(t) - \imath_i(t))\, R(t)\ \mathrm{d}t$$
$$+ \int_0^\infty \imath_i(t) \int_t^{t+J_i(t)} (\phi_i(\tau) - w(\tau)\, q(t))\, R(\tau) \mathrm{d}\tau\ \mathrm{d}t$$
$$+ \int_{-T_i(0)}^0 \imath_i(t) \int_0^{t+J_i(t)} (\phi_i(\tau) - w(\tau)\, q(t))\, R(\tau)\, \mathrm{d}\tau\ \mathrm{d}t$$

where $\phi_i(t)$ represents the shadow value of $y_i(t)$ and $J_i(t) = T_i(t + J_i(t))$ (notice that $T_i(t) = J_i(t - T_i(t))$). The first order conditions with respect to $y_i(t)$, $\imath_i(t)$ and $J_i(t)$ are respectively, $\forall\, t \geq 0$

$$\phi_i(t) = \left(1 - \frac{1}{\epsilon}\right) p_i(t)$$

$$R(t) = \int_t^{t+J_i(t)} (\phi_i(\tau) - w(\tau)\, q(t))\ R(\tau)\ \mathrm{d}\tau$$

and $\forall\, t \geq -T_i(0)$

$$\phi(t + J_i(t)) = w(t + J_i(t))\, q(t).$$

At the symmetric equilibrium, $p_i(t) = 1$, $y_i(t) = y(t)$, $l_i(t) = l(t)$, $J_i(t) = J(t)$, $T_i(t) = T(t)$, $\phi_i(t) = \phi(t)$ and $\imath_i(t) = \imath(t)$. In which case, $\forall t \geq 0$:

$$\phi(t) = \left(1 - \frac{1}{\epsilon}\right) \equiv \mu,$$

$$R(t) = \int_t^{t+J(t)} (\mu - w(\tau) q(t))\, R(\tau)\ \mathrm{d}\tau$$

$$w(t)\, q(t - T(t)) = \mu$$

Notice that parameter $0 < \mu < 1$, since $\epsilon > 1$.

3 Decentralized equilibrium

From previous sections, the equilibrium of this economy is characterized by the following equation system, $\forall t \geq 0$:

$$y(t) = \int_{t-T(t)}^t \imath(\tau)\ \mathrm{d}\tau \tag{7}$$

$$1 = \int_{t-T(t)}^t q(\tau)\, \imath(\tau)\ \mathrm{d}\tau \tag{8}$$

$$y(t) = c(t) + \imath(t) \tag{9}$$

$$\exp\{-\rho t\} = \int_t^{t+J(t)} (\mu - w(\tau)\, q(t)) \exp\{-\rho\tau\}\ \mathrm{d}\tau \tag{10}$$

$$w(t)\, q(t - T(t)) = \mu \tag{11}$$

$$J(t) = T(t + J(t)) \tag{12}$$

$$q(t) = q_0 \exp\{-\gamma t\} \tag{13}$$

with initial conditions

$$i_t \ \forall \ t < 0 \ \text{given.}$$

Equation (8) represents the equilibrium in the labor market, where labor supply is normalized to one. Equation (9) represents the equilibrium in the goods market. Finally, in equation (13) the parameter $\gamma > 0$ represents (Harrod neutral) technical progress. All other equations were previously derived from agents' problems.

Equations (7), (8), (10), (11) and (12) allow us to solve for the endogenous variables $y(t)$, $i(t)$, $w(t)$, $T(t)$ and $J(t)$, given the exogenous technological process described by equation (13). Using equation (9), we can solve for $c(t)$, which is undetermined in the consumer problem. In the next section, we will show that the forward-looking component of the model, summarized in equations (10) to (13), determines the equilibrium scrapping rule $T(t), \forall t \geq 0$.

Equilibrium Scrapping Rule

Notice that equations (10) to (13) form a block, where $T(t)$ can be solved as a function of $J(t)$, the exogenous process $q(t)$ and the discount factor ρ. The scrapping rule $T(t)$ is forward-looking, but it depends on its own value in a particular and endogenous point of time, i.e., $J(t) = T(t + J(t))$. This type of variable is not standard in economic models and we are concerned with the solution of this non-standard dynamic problem.

Let us differentiate equation (10) and rearrange terms,

$$\mu - (q(t)w(t) + \rho) = -\dot{q}(t) \exp\{\rho t\} \int_t^{t+J(t)} w(\tau) \exp\{-\rho\tau\}\ \mathrm{d}\tau \tag{14}$$

As pointed out by Malcomson, the left hand side of this condition represents the difference between the marginal benefit, given by μ, and operational and capital costs of producing on period t with the new machines. The right hand side represents the opportunity cost of investing today and not tomorrow, given that subsequent vintages have lower operational costs.

From equations (10) and (11), we can rewrite (14), $\forall t \geq 0$, as

$$\exp\{-\gamma T(t)\} = 1 - \frac{\rho - \gamma}{\mu} - \frac{\gamma}{\rho} + \frac{\gamma}{\rho} \exp\{-\rho J(t)\}, \tag{15}$$

which is a forward-looking condition for $T(t)$ -as a function of its own future value $T(t+J(t)) = J(t)$. The linear utility assumption, which implies that the interest rate is constant over time (and equal to ρ), allows us to characterize the forward-looking equation for the scrapping rule $T(t)$ as a function of the expected scrapping of contemporaneous replacement investment $J(t)$ alone.[2]

The scrapping function $T(t)$ and the associated function $J(t)$ are related by equations (12) and (15). Analogous problems had been handled by Malcomson (1975) and van Hilten (1991), within a partial equilibrium framework. In particular, van Hilten presents a sufficient condition allowing to check for the so-called Terborgh-Smith result, i.e., $T(t)$ is constant. The major outline of van Hilten's proof consist in isolating a function $F(.)$ such that $T(t) = F(J(t))$ for any $T(t) \geq 0$ and $J(t) \geq 0$. An equation similar to (15) is used to this end. As $T(t)$ and $J(t)$ verify $J(t) = T(t+J(t)), \forall t \geq 0$, a fixed-point argument is then built up and can be applied successfully if function $F(.)$ is well-behaved (more precisely, if it is increasing and admitting a unique fixed-point). In our case, the construction of an analogous function $F(.)$ from equation (15) requires some restrictions on the parameters ρ, γ and μ according to the following proposition:

Proposition 1 *Equation (15) determines a function $F(.)$ such that: $T(t) = F(J(t))$ and $F(.)$ well defined for any $T(t) \geq 0$ and $J(t) \geq 0$, if and only if:*

$$1 > \mu \geq \rho > \gamma > 0. \tag{16}$$

Proof: See the appendix for a proof.

Condition (16) allows to determine a function $F(.)$ such that: $T(t) = F(J(t))$ and $F(.)$ well defined for any $T(t) \geq 0$ and $J(t) \geq 0$. That is:

$$F(x) = -\frac{1}{\gamma} \ln \left\{ 1 - \frac{\rho - \gamma}{\mu} - \frac{\gamma}{\rho} [1 - \exp\{-\rho x\}] \right\} \tag{17}$$

for any $x \geq 0$.

Under (16), we can also check van Hilten's requirements on function $F(.)$, that is:

$$F'(x) > 0, \quad \forall x \geq 0 \tag{18}$$

$$F(x) \text{ admits a unique strictly positive fixed-point} \tag{19}$$

Consequently, we can also state the Terborgh-Smith property within our setting:

[2]In the non-linear utility case, the interest rate is endogenous and depends on the scrapping rule $T(t)$. The economy is then characterized by a differential-difference system with endogenous $T(t)$ and the equilibrium can not be solved by Bellman and Cooke (1963) technique as in Benhabib and Rustichini (1993).

Proposition 2 *For any parameters ρ, γ and μ checking condition (16), there exists a unique constant $T_F(\rho,\gamma,\mu)$ such that:*

- $T(t) = T_F(\rho,\gamma,\mu), \forall t \geq 0$, *is the unique solution of equation (15);*
- $F(T_F(\rho,\gamma,\mu)) = T_F(\rho,\gamma,\mu) : T_F(\rho,\gamma,\mu)$ *is the fixed-point of the function* $F(.)$

Proof: See the appendix for a proof.

The function $F(.)$ defined in (17) is represented in Figure 1. This picture is equivalent to a standard phase diagram for a first-order difference equation. The function $F(.)$ intersects the diagonal in the unique strictly positive fixed-point, which is forward-looking stable.[3] As in standard rational expectations models, Proposition 2 states that $T(t)$ must be constant and equal to the fixed-point of the stable forward-looking relation. Notice that it is the standard "rational expectation principle" for a saddle-path solution in dynamic general equilibrium models with forward-looking variables.

As in the standard growth model, the next proposition states that wages grow at the same rate as productivity.

Proposition 3 *For any parameters ρ, γ and μ checking condition (16), the dynamic behavior of wages is given by, $\forall t \geq 0$:*

$$\frac{\dot{w}(t)}{w(t)} = \gamma \tag{20}$$

Proof: By Proposition 2, we know that the scraping function is constant. Condition (20) is obtained by differentiating equation (11) and rearranging terms. □

Dynamics of Production, Investment and Wages

As $T(t) = T_F, \forall t \geq 0$, equation (8) should determine $\imath(t)$. However, it is clear that the latter implies a mathematically weak determination, $\imath(t)$ only appearing as an integrand of a Riemann integral. At least, we need a continuity assumption to get an explicit solution for investment. Since the investment solution $\imath(t)$, $t \geq 0$, by equation (8) depends strongly upon the characteristics of the investment initial path, $\imath(t)$, $t < 0$, we assume that it is piecewise continuous and we look for piecewise continuous investment solution paths.

The time-independence of the scrapping rule implies that the relevant investment initial conditions have to be taken on the time interval $[-T_F, 0[$. We state now our piecewise continuity requirement **PCR** on the initial path:

[3]It can be easily show that $0 < F'(x) < 1$, $\forall x \geq 0$.

Assumption 1 *(**PCR**) The initial investment path, $\imath(t)$ with $t \in [-T_F, 0[$, checks*

$$\begin{array}{ll} \exists & (t_0 = -T_F < t_1 < \ldots < t_n = 0) \in \Re_-^{n+1} \quad \textit{such that} \\ i) & \imath(t) \quad \textit{is continuous on} \quad]t_j, t_{j+1}[\quad \forall j, 0 \leq j \leq n-1 \\ ii) & \forall j, 0 < j \leq n, \lim_{t \to t_j^-} \imath(t) < +\infty \\ & \forall j, 0 \leq j < n, \lim_{t \to t_j^+} \imath(t) < +\infty \end{array}$$

Moreover, since we are interested in solutions for which the scrapping rule is constant, the labor market equilibrium condition (8) must hold with $T(t) = T_F, \forall t \geq 0$. To make sure that the labor market is at equilibrium at the initial period with $T(0) = T_F$, we impose the following assumption on investment initial conditions:

Assumption 2 *The initial investment path, $\imath(t)$ with $t \in [-T_F, 0[$, checks*

$$\int_{-T_F}^{0} q(\tau)\,\imath(\tau)\ d\tau = 1. \tag{21}$$

This assumption does not seem to be very restrictive since we are concerned with the analysis of business cycles. Observe that it is just a normalization of the investment initial conditions, but it does not restrict at all the nature of the historical dynamics.[4] Finally, we impose the following assumption to be sure that consumption is positive, or equivalently that investment is lower than production:

Assumption 3 *The initial investment path, $\imath(t)$, $t < 0$, checks*

$$\imath(t) \leq \int_{-T_F}^{t} \imath(\tau)\, d\tau + \exp\{-\gamma T_F\} \int_{t}^{0} \imath(\tau)\, d\tau$$

$$\forall\, t \in [-T_F, 0[,\ t \neq t_j,\ 0 \leq j \leq n.$$

We can now prove the main proposition of the paper:

[4]By Assumption 2 we set the initial level of the capital stock, by allowing the economy to full-employ workers at $t = 0$ with just the first T_F generations of machines. In the general case the initial stock of machines could be relatively high or low, with respect to Assumption 2. If it is relatively high (Resp. low) investment must be zero (Resp. equal to output) over a finite interval until condition

$$\int_{t_0 - T_F}^{t_0} q(\tau)\,\imath(\tau)\ d\tau = 1.$$

holds for some $t_0 > 0$. Then, beginning at $t = t_0$ we get the solution developed in the paper.

Proposition 4 *For any parameters ρ, γ and μ checking condition (16), and for any initial conditions checking Assumptions 1 and 2, the solution path $\imath(t)$, $t \geq 0$ must check:*

$P1)$ $\imath(t)$ *is piecewise continuous:*

$$\forall k \in \mathcal{N},\ \imath(t) \textit{ is continuous on }]t_j + (k+1)T_F, t_{j+1} + (k+1)T_F[$$

$$\forall j, 0 \leq j \leq n-1$$

$P2)$ $\forall t \geq 0\ \ t \neq t_j + (k+1)T_F\ \ \forall j, 0 \leq j \leq n$ *and* $k \in \mathcal{N}$:

$$\imath(t) = \imath(t - T_F)\exp\{\gamma T_F\} \tag{22}$$

$P3)$ $\forall j,\ 0 < j \leq n-1,\ \forall k \in \mathcal{N}:\qquad \lim_{t \to (t_j+(k+1)T_F)^-} \imath(t) < +\infty$

$\forall j,\ 0 \leq j < n-1,\ \forall k \in \mathcal{N}:\qquad \lim_{t \to (t_j+(k+1)T_F)^+} \imath(t) < +\infty$

Proof: See the Appendix for a proof.

Investment is piecewise continuous and its dynamic behavior is characterized by the difference equation (22). The computation of the solution path for investment requires initial conditions over the whole time interval $[-T_F, 0[$ (refered as "historical conditions" in the Introduction). As T_F is endogenously determined by the forward-looking component of the scrapping rule, the later proposition points out the major outline of the model: expectations entirely determine the length of initial conditions, but the nature of the system's dynamics matches the pattern of the relevant initial conditions.

Investment follows productivity, with an exponential trend equal to γ, and presents a purely periodic behavior of length T_F around this trend. The profile of the business fluctuations depends crucially on initial conditions over the interval $[-T_F, 0[$. Historical conditions are relevant because technical progress is embodied in machines. When at period t the scrapped capital is high (with respect to the trend), replacement investment is also high to insure full-employment inducing an increase of labor productivity greater than the trend.[5]

Until now no sign restrictions were imposed over investment and consumption. From (22) we know that $\imath(t) \geq 0, \forall t \geq 0$ if $\imath(t) \geq 0, \forall t < 0$. A positive path for consumption requires Assumption 3.

Concerning production we can state the following proposition:

Proposition 5 *For any parameters ρ, γ and μ checking condition (16), and for any initial conditions checking Assumptions 1 and 2, the dynamic behavior of production*

[5]Notice that historical conditions do not matter in the standard neoclassical growth model because capital is homogeneous.

is given by: $\forall t \in [0, T_F[$

$$y(t) = \exp\{\gamma T_F\} \int_{-T_F}^{t-T_F} \imath(\tau)\, d\tau + \int_{t-T_F}^{0} \imath(\tau)\, d\tau.$$

and $\forall\, t \geq T_F$

$$y(t) = y(t - T_F) \exp\{\gamma T_F\}. \tag{23}$$

Proof: By Proposition 2, we know that the scraping function is constant and equal to T_F. Conditions for $y(t)$ are obtained by substituting condition (22) in equation (7) and rearranging terms. □

The pattern of output in the time interval $[0, T_F[$ is determined by initial conditions on investment. Afterwards, output is characterized by the difference equation (23) and its path reproduces the pattern of the time interval $[0, T_F[$.

Concluding Remarks

In a general equilibrium framework, we analyze the dynamic properties of the vintage capital hypothesis. Given the complexities of the dynamic system, we restrict ourselves to the analysis of the linear utility case with stationary environment. Under these assumptions, we show the time independence of the scrapping rule and the existence of a purely periodic behavior of output around an exponential trend. In this sense, this paper provide a new view on fluctuations: they depend crucially on historical conditions. In the real world, fluctuations depend also on other phenomena as non-linearities and an stochastic environment.

Some extensions seem natural, even if they are technically complex. The first one is to extend the model to non-stationary environments, in order to analyze how cyclical patterns on preferences or technical progress can affect the scrapping rule and the dynamic behavior of output. The hypothesis is that the dynamic behavior of production depends on the pattern of initial conditions and on the dynamic behavior of the environment.

In a second step, we would like to generalize our result to a non-linear utility environment. In which case, the interest rate becomes endogenous and the scrapping rule can not be isolated from the rest of the dynamic system. The problem becomes mainly technical, since we must solve a differential-difference system with endogenous leads and lags.

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Appendix: Propositions' Proof

Proposition 1

From equation (15), $T(t) \geq 0$ requires:

$$0 \leq 1 - \frac{\rho - \gamma}{\mu} - \frac{\gamma}{\rho}[1 - \exp\{-\rho J(t)\}] \leq 1$$

If $J(t)$ is allowed to take any positive value, the parameters ρ, γ and μ must check:

$$1 - \frac{\rho - \gamma}{\mu} - \frac{\gamma}{\rho} \geq 0$$

$$1 - \frac{\rho - \gamma}{\mu} \leq 1$$

which yields $1 > \mu \geq \rho \geq \gamma > 0$.

Later, we rule out the special case $\rho = \gamma$, from which equation (15) becomes:

$$\exp\{-\gamma T(t)\} = \exp\{-\gamma J(t)\}$$

and then $T(t) = J(t) \, \forall t \geq 0$. The constraint (12), $\forall t \geq 0$ is consequently equivalent to :

$$T(t) = T(t + T(t)), \forall t \geq 0.$$

It is clear that any constant function $T(t)$ checks the later equation: T(t) will not be *uniquely* determined by (15). □

Proposition 2

Since $J(t) \geq 0$ $\forall t \geq 0$ and $F'(x) > 0$, we have $\forall t \geq 0$:

$$F(0) \leq T(t) \leq \bar{F}$$

with $\bar{F} = \lim_{x \to \infty} F(x)$

Applying the latter inequalities at $t + J(t)$, using (12) we get $\forall t \geq 0$:

$$F(0) \leq J(t) \leq \bar{F}$$

By property (18), it yields $\forall t \geq 0$:

$$F\left(F(0)\right) \leq T(t) \leq F\left(\bar{F}\right).$$

As before, we can apply the latter inequalities at $t + J(t)$, find new lower and upper bound for $J(t)$ and establish, by property (18), new lower and upper bound for $T(t)$, $\forall t \geq 0$. Repeating this reasoning, we can construct a sequence of lower bounds X_n and upper bounds Y_n for $T(t)$, $\forall t \geq 0$, such that,

$$\begin{array}{lll} \forall n \geq 0 \text{ and } t \geq 0 & X_n \leq T(t) \leq Y_n; & \\ X_0 = F(0), & \text{and } X_n = F(X_{n-1}) & \forall n \geq 1; \\ Y_0 = \bar{F}, & \text{and } Y_n = F(Y_{n-1}) & \forall n \geq 1. \end{array}$$

Trivially, $Y_0 > Y_1$ and $X_0 < X_1$. $F(.)$ being strictly increasing, the sequence $\{Y_n\}_{n \geq 0}$ is decreasing and $\{X_n\}_{n \geq 0}$ is increasing. Finally, observe that $X_0 \leq Y_n \leq Y_0$ and $X_0 \leq X_n \leq Y_0$, $\forall n \geq 0$; the two sequences are bounded. Since they are monotonic, both of them converge and, by construction of these sequences, the limits are equal to the unique fixed-point of function $F(.)$. □

Proposition 4

Proving the statements P1 to P3 on $[0, T_F[$ is sufficient, since a simple time translation ensures them on the whole interval $[0, +\infty[$. From (8), $\forall t \in [0, T_F[$,

$$\int_{t-T_F}^{t} i(\tau) q(\tau) \, \mathrm{d}\tau = 1.$$

So:

$$\int_{0}^{t} i(\tau) q(\tau) \, \mathrm{d}\tau = 1 - \int_{t-T_F}^{0} i(\tau) q(\tau) \, \mathrm{d}\tau.$$

Furthermore, $\exists j, 0 \leq j \leq n$, such that:

$$T_F + t_j \leq t \leq T_F + t_{j+1}.$$

Hence, we can write the previous equation as

$$\int_0^t i(\tau)q(\tau)\,\mathrm{d}\tau = 1 - \int_{t-T_F}^{t_{j+1}} i(\tau)q(\tau)\,\mathrm{d}\tau - \int_{t_{j+1}}^{0} i(\tau)q(\tau)\,\mathrm{d}\tau.$$

Let us assume, as in statement P2, that: $T_F + t_j < t < T_F + t_{j+1}$. By continuity of $i(t)$ on $]t_j, tj+1[$,

$$\int_{t-T_F}^{t_{j+1}} i(\tau)q(\tau)\,\mathrm{d}\tau$$

is differentiable on this interval and so is $\int_0^t i(\tau)q(\tau)\,\mathrm{d}\tau$ given the later equation. Differentiation leads: $i(t) = i(t-T_F)\exp\{\gamma T_F\}$. Taking the limits of the latter equation at the bounds of the interval ensures property P3. Property P1 derives from P2.

It remains to be shown that $i(t)$'s solution given by P2 checks equation (8), $\forall t \geq 0$. As before, we will show it for any t in the interval $[0, T_F[$. Notice that for a well chosen j, t must also be in $]t_j + T_F, t_{j+1} + T_F[$, allowing us to write (let us assume $q_0 = 1$):

$$\begin{aligned}
\int_{t-T_F}^{t} i(\tau)q(\tau)\,\mathrm{d}\tau &= \textstyle\int_{t-T_F}^{t} i(\tau)\exp\{-\gamma\tau\}\,\mathrm{d}\tau \\
&= \textstyle\int_{t-T_F}^{0} i(\tau)\exp\{-\gamma\tau\}\,\mathrm{d}\tau + \displaystyle\int_0^{t_j+T_F} i(\tau)\exp\{-\gamma\tau\}\,\mathrm{d}\tau \\
&\quad + \int_{t_j+T_F}^{t} i(\tau)\exp\{-\gamma\tau\}\,\mathrm{d}\tau.
\end{aligned}$$

Then using the relation: $i(t) = i(t-T_F)\exp\{\gamma T_F\}$, $\forall t \geq 0$ and $t \neq t_j + T_f \forall j$, we get

$$\begin{aligned}
\int_{t-T_F}^{t} i(\tau)q(\tau)\,\mathrm{d}\tau &= \textstyle\int_{t-T_F}^{0} i(\tau)\exp\{-\gamma\tau\}\,\mathrm{d}\tau + \displaystyle\int_0^{t_j+T_F} i(\tau-T_F)\exp\{-\gamma(\tau-T_F)\}\,\mathrm{d}\tau \\
&\quad \int_{t_j+T_F}^{t} i(\tau-T_F)\exp\{-\gamma(\tau-T_F)\}\,\mathrm{d}\tau.
\end{aligned}$$

An elementary variable change yields:

$$\begin{aligned}
\int_{t-T_F}^{t} i(\tau)q(\tau)\,\mathrm{d}\tau &= \textstyle\int_{t-T_F}^{0} i(\tau)\exp\{-\gamma\tau\}\,\mathrm{d}\tau + \displaystyle\int_{-T_F}^{t_j} i(\tau)\exp\{-\gamma\tau\}\,\mathrm{d}\tau \\
&\quad + \int_{t_j}^{t-T_F} i(\tau)\exp\{-\gamma\tau\}\,\mathrm{d}\tau.
\end{aligned}$$

Hence

$$\int_{t-T_F}^{t} i(\tau)q(\tau)\,\mathrm{d}\tau = \int_{-T_F}^{0} i(\tau)q(\tau)\,\mathrm{d}\tau = 1.\Box$$

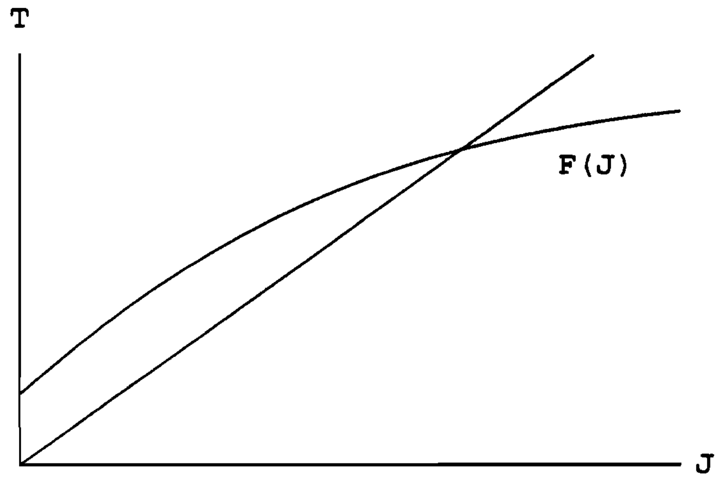

Figure 1: The scrapping rule